

THEORY OF MUONIC HYDROGEN - MUONIC DEUTERIUM ISOTOPE SHIFT

A.P. Martynenko*

*Samara State University,
443011, Ac. Pavlov 1, Samara, Russia*

We calculate the corrections of orders α^3 , α^4 and α^5 to the Lamb shift of the $1S$ and $2S$ energy levels of muonic hydrogen (μp) and muonic deuterium (μd). The nuclear structure effects are taken into account in terms of the proton r_p and deuteron r_d charge radii for the one-photon interaction and by means of the proton and deuteron electromagnetic form factors in the case of one-loop amplitudes. The obtained numerical value of the isotope shift (μd) - (μp) for the splitting ($1S \div 2S$) 101003.3495 meV can be considered as a reliable estimation for corresponding experiment with the accuracy 10^{-6} . The fine structure interval $E(1S) - 8E(2S)$ in muonic hydrogen and muonic deuterium are calculated.

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I. INTRODUCTION

The investigation of the energy spectra of hydrogenic atoms is of great importance for high accuracy verification of the Standard Model and derivation more correct values of fundamental physical constants (the fine structure constant, the masses of the muon and electron, the proton charge radius etc.) [1, 2, 3]. In the last few years significant interest in this research area is connected with the atom of muonic hydrogen [4, 5, 6, 7]. It has motivated by the activity of the experimental study of the ($2P - 2S$) Lamb shift and hyperfine structure in muonic hydrogen. The measurement of the ($2P - 2S$) Lamb shift with the accuracy 30 ppm in the (μp) would allow us to obtain the value of the proton charge radius with a precision 10^{-3} which is an order of the magnitude better in the comparison with different methods including the elastic electron-proton scattering and the value of the ($2P - 2S$) Lamb shift in the atom of electronic hydrogen. The measurement of the ground state hyperfine splitting in muonic hydrogen with a similar accuracy would allow to determine the value of the Zemach radius [8] with a precision 10^{-3} [9, 10] which can be considered as a new fundamental parameter regarding to the hydrogen atom. Then it can be used in the calculation of new theoretical quantity of the hyperfine splitting in electronic hydrogen and the restriction on the value of the proton polarizability contribution [11, 12, 13, 14].

There exist another experimental task of the investigation of large fine structure interval ($1S - 2S$) in muonic hydrogen and the muonic hydrogen - muonic deuterium isotope shift

*Electronic address: mart@ssu.samara.ru(A.P.Martynenko)

for this splitting [15, 16]. It would allow to obtain new data about the proton and deuteron charge radii. It should be pointed out that both marked intervals are among the most precise measured quantities for the atom of electronic hydrogen. Experimental accuracy for the measurement of the hydrogen-deuterium isotope shift for the transition ($1S - 2S$) increased by three orders of the magnitude during last ten years. At present time it is equal [17]

$$\Delta\nu_{IS} = [E(2S) - E(1S)]_D - [E(2S) - E(1S)]_H = 670\ 994\ 334.64(15) \text{ kHz}. \quad (1)$$

The interval ($1S - 2S$) for the hydrogen atom was measured with extremely high accuracy of order of 10^{-2} kHz [18]:

$$\Delta\nu_{1S-2S}(H) = 2\ 466\ 061\ 413\ 187\ 103(46) \text{ Hz}, \quad \delta = 1.8 \cdot 10^{-14}. \quad (2)$$

Experimental investigations of the energy intervals (1) and (2) in muonic hydrogen are on the stage of preliminary preparation.

Theoretical studies of various contributions to the energy levels of muonic atoms were made long ago [19, 20, 21] (see also the Refs. in the review article [1]). In the last years the calculation of different order corrections in the energy spectrum of muonic hydrogen was connected for the most part with the ($2P - 2S$) Lamb shift and hyperfine structure of S -states [13, 22, 23, 24, 25]. In these papers the two-particle interaction operator was constructed which gave the contributions of orders α^5 and α^6 to the interval ($2P - 2S$) and hyperfine splitting of the $1S$ and $2S$ states. At present the necessity exists for theoretical investigation of the corrections of orders α^3 , α^4 and α^5 to the $1S$ and $2S$ Lamb shift in muonic hydrogen and muonic deuterium, the isotope shift (μp) - (μd) for the transition ($1S - 2S$) and the fine structure interval [$E(1S) - 8E(2S)$] which are yet unknown. Such calculations can initiate the experimental investigations of the fine structure intervals (1) and (2) in muonic hydrogen in order to obtain more exact values of such fundamental physical constants as the proton and deuteron charge radii and the muon mass.

In this paper we have derived numerical results for the contributions of orders α^3 , α^4 and α^5 to the splitting ($1S - 2S$) and the isotope shift (μp) - (μd) for this splitting. In doing so, we obtained numerical values for some contributions on the basis of known analytical expressions. The most part of the corrections for the energy levels $1S$ and $2S$ and the isotope shift in muonic hydrogen is determined originally in the integral form which then is used for numerical estimations. The dependence on principal quantum number n for the most part of the contributions is not trivial that is not reduces to the factor $1/n^3$. The reason has to do with characteristic value of the particle momenta and will be discussed below. The aim of this work consists in the calculation of quantum electrodynamic corrections in the muonic hydrogen - muonic deuterium isotope shift for the transition ($1S - 2S$) and the fine structure interval [$E(1S) - 8E(2S)$] both for (μp) and (μd) with the accuracy 10^{-9} which can be considered as a proper guide for the future experiments and the retrieval of the information about the proton and deuteron charge radii and the muon mass.

Fine structure of the energy levels of hydrogenic atoms has been studied for a long time on the basis of different approaches [1, 16, 26, 27]. For the case of S -states of hydrogen-like atoms consisting of the particles with masses m_1 and m_2 it can be written with the accuracy $O((Z\alpha)^4)$ as follows:

$$E_n = m_1 + m_2 - \frac{\mu^2(Z\alpha)^2}{2n^2} - \frac{\mu(Z\alpha)^4}{2n^3} \left[1 - \frac{3}{4n} + \frac{\mu^2}{4m_1 m_2 n} \right] = \quad (3)$$

$$= \begin{cases} \mu p(1S) : 1\ 043\ 927\ 826\ 470.3586 \text{ meV}; \mu p(2S) : 1\ 043\ 929\ 722\ 866.0601 \text{ meV} \\ \mu d(1S) : 1\ 981\ 268\ 455\ 762.7537 \text{ meV}; \mu d(2S) : 1\ 981\ 270\ 453\ 188.8081 \text{ meV} \end{cases}.$$

The error of relative order in Eq.(3) which is determined by the uncertainties of the fine structure constant α and the particle masses reaches the value 10^{-7} . Nevertheless, we represented the numerical values in Eq.(3) with the precision 0.0001 meV because it is important when we consider various fine structure intervals in the energy spectrum. The following values of fundamental physical constants are used: $\alpha^{-1} = 137.03599976(50)$, $m_\mu = 0.105658357(5)$ GeV, $m_p = 0.938271998(38)$ GeV, $m_d = 1.875612762(75)$ GeV [3]. The contribution of Eq.(3) to the (μp) - (μd) isotope shift for the splitting $(1S - 2S)$ is the crucial (see the Table I). In addition, there exist a number of important effects both electromagnetic and strong interactions which must be taken into account for the determination of the isotope shift value with the accuracy up to terms of order α^5 .

II. EFFECTS OF ONE-LOOP AND TWO-LOOP VACUUM POLARIZATION IN ONE-PHOTON INTERACTION

Our calculations of various energy levels of hydrogen-like atoms are performed within the framework of the quasipotential approach in which a bound state of two particles is described by the Shroedinger-type equation [28, 29]:

$$[G^f]^{-1} \psi_M \equiv \left(\frac{b^2}{2\mu_R} - \frac{\mathbf{p}^2}{2\mu_R} \right) \psi_M(\mathbf{p}) = \int \frac{d\mathbf{q}}{(2\pi)^3} V(\mathbf{p}, \mathbf{q}, M) \psi_M(\mathbf{q}), \quad (4)$$

where

$$b^2 = E_1^2 - m_1^2 = E_2^2 - m_2^2,$$

$\mu_R = E_1 E_2 / M$ is the relativistic reduced mass, $M = E_1 + E_2$ is the mass of the bound state. The quasipotential in Eq.(4) is constructed in quantum electrodynamics by the perturbation theory with the use of the two-particle scattering amplitude T projected onto the positive frequency states outside the mass surface at zero relative energies of the particles:

$$V = V^{(1)} + V^{(2)} + V^{(3)} + \dots, \quad T = T^{(1)} + T^{(2)} + T^{(3)} + \dots, \quad (5)$$

$$V^{(1)} = T^{(1)}, \quad V^{(2)} = T^{(2)} - T^{(1)} \times G^f \times T^{(1)}, \dots . \quad (6)$$

The initial approximation of the quasipotential $V(\mathbf{p}, \mathbf{q}, M)$ for a bound system was selected in the form of the usual Coulomb potential. The increase in the lepton mass in muonic hydrogen compared with its electronic counterpart decreases the radius of the Bohr orbit in the (μp) . As a result, the Compton wave length of the electron and the radius of the Bohr orbit become commensurable:

$$\frac{\hbar^2}{\mu e^2} : \frac{\hbar}{m_e c} = 0.737384, \quad (7)$$

where m_e is the mass of the electron, μ is the reduced mass in the (μp) atom. This substantially enhances the role played by vacuum polarization effects in the energy spectrum of muonic hydrogen [30]. Corrections of the one-loop and two-loop vacuum polarization to the quasipotential of one-photon interaction are shown in Fig.1.

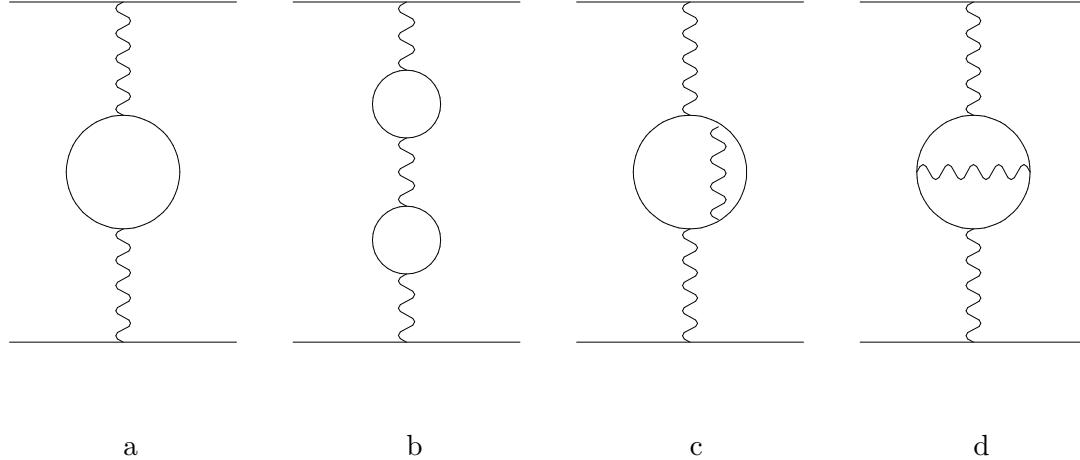


FIG. 1: Effects of one-loop and two-loop vacuum polarization in one-photon interaction.

To determine the contribution of diagram (a) in Fig.1 (electronic vacuum polarization) to the particle interaction operator we must perform the following substitution in the photon propagator [30]:

$$\frac{1}{k^2} \rightarrow \frac{\alpha}{3\pi} \int_1^\infty ds \frac{\sqrt{s^2 - 1}(2s^2 + 1)}{s^4(k^2 + 4m_e^2 s^2)}. \quad (8)$$

If

$$(-k^2) = \mathbf{k}^2 \sim \mu_e^2(Z\alpha)^2 \sim m_e^2(Z\alpha)^2$$

(electronic hydrogen, μ_e is the reduced mass of two particles in the hydrogen atom), then, ignoring the first term in the denominator in the right-hand side of Eq.(8) we obtain:

$$-\alpha/15\pi m_e^2.$$

However, if

$$\mathbf{k}^2 \sim \mu^2(Z\alpha)^2 \sim m_1^2(Z\alpha)^2$$

(muonic hydrogen, m_1 is the mass of the muon), then $\mu\alpha$ and m_e are values of one order and we cannot expand the denominator in Eq.(8) in α . With muonic hydrogen, we must construct the particle interaction operator in the one-photon approximation using exact expression (8). Further, we take into account that the electron vacuum polarization gives the contributions of orders α^3 , α^4 and α^5 to the energy spectrum of S -states in the (μp) atom.

The modification of the Coulomb potential

$$V^C(\mathbf{k}) = -Ze^2/\mathbf{k}^2$$

caused by the vacuum polarization (VP) is determined taking into account (8) by the following expression in the momentum representation [30]:

$$V_{VP}^C(\mathbf{k}) = -4\pi Z\alpha \frac{\alpha}{\pi} \int_1^\infty \frac{\sqrt{\xi^2 - 1}}{3\xi^4} \frac{(2\xi^2 + 1)}{\mathbf{k}^2 + 4m_e^2 \xi^2} d\xi \quad (9)$$

The Fourier transform of Eq.(9) gives the corresponding operator in the coordinate representation:

$$V_{VP}^C(r) = \frac{\alpha}{3\pi} \int_1^\infty d\xi \frac{\sqrt{\xi^2 - 1}(2\xi^2 + 1)}{\xi^4} \left(-\frac{Z\alpha}{r} e^{-2m_e \xi r} \right). \quad (10)$$

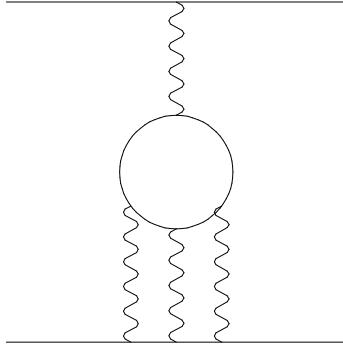


FIG. 2: The Wichmann-Kroll correction. The wave line denotes the Coulomb photon.

The potential (10) allows us to obtain the correction of the electron vacuum polarization of order α^3 to the energy levels of $1S$ and $2S$ -states in muonic hydrogen. Accounting exact expressions of the wave functions for $1S$ and $2S$ -states

$$\psi_{100}(r) = \frac{W^{3/2}}{\sqrt{\pi}} e^{-Wr}, \quad \psi_{200}(r) = \frac{W^{3/2}}{2\sqrt{2\pi}} e^{-Wr/2} \left(1 - \frac{Wr}{2}\right), \quad W = \mu Z\alpha, \quad (11)$$

we represent this correction in the form:

$$\Delta E_{1\gamma,VP}(1S) = -\frac{\mu(Z\alpha)^2\alpha}{3\pi} \int_1^\infty \rho(\xi) d\xi \frac{1}{p_1^2(\xi)}, \quad p_1(\xi) = 1 + \frac{m_e\xi}{W}, \quad \rho(\xi) = \frac{\sqrt{\xi^2 - 1}(2\xi^2 + 1)}{\xi^4}, \quad (12)$$

$$\Delta E_{1\gamma,VP}(2S) = -\frac{\mu(Z\alpha)^2\alpha}{6\pi} \int_1^\infty \rho(\xi) d\xi \left(\frac{1}{p_2^2(\xi)} - \frac{2}{p_2^3(\xi)} + \frac{3}{2p_2^4(\xi)}\right), \quad p_2(\xi) = 1 + \frac{2m_e\xi}{W}. \quad (13)$$

The electron vacuum polarization effects are very sensitive to the bound state structure because the characteristic momentum of the particles in muonic hydrogen atom is $(\mu Z\alpha)$. So, the contribution of the vacuum polarization amplitudes in one-photon interaction cannot be expressed by a simple factor $|\psi^C(0)|^2 \sim 1/n^3$ but it is dependent on principal quantum number n in more complicated form which is represented by expressions (12) and (13). Numerical values of the electron vacuum polarization contribution to the $1S$ and $2S$ Lamb shift in muonic hydrogen and muonic deuterium differ only due to reduced mass of two particles:

$$\Delta E_{1\gamma,VP} = \begin{cases} \mu p(1S) : -1898.8379 \text{ meV}; \mu p(2S) : -219.5849 \text{ meV} \\ \mu d(1S) : -2129.2820 \text{ meV}; \mu d(2S) : -245.3205 \text{ meV} \end{cases} \quad (14)$$

The contribution of the muon vacuum polarization (MVP) can be found also with the use of Eq.(9) in which the replacement $m_e \rightarrow m_1$ must be carried out. This correction of order α^5 to the energy spectrum is included in Table I together with the muon self energy correction (MSE).

The diagram in Fig.2 has the form of one-loop vacuum polarization. It determines the correction to the energy spectrum of order α^5 known as the Wichmann-Kroll correction [31, 32]. The corresponding potential was represented in the integral form:

$$\Delta V^{WK}(r) = \frac{\alpha(Z\alpha)^3}{\pi r} \int_0^\infty \frac{d\zeta}{\zeta^4} e^{-2m_e\zeta r} \left[-\frac{\pi^2}{12} \sqrt{\zeta^2 - 1} \theta(\zeta - 1) + \int_0^\zeta dx \sqrt{\zeta^2 - x^2} f^{WK}(x) \right]. \quad (15)$$

Exact expression for the spectral function f^{WK} is written in Ref.[1, 31, 32]. Averaging the potential (15) over the Coulomb wave functions (11) we expressed the corrections to the S -states as follows:

$$\Delta E^{WK}(1S) = \frac{\mu\alpha(Z\alpha)^4}{\pi} \int_0^\infty \frac{d\zeta}{\zeta^4 p_1^2(\zeta)} \left[-\frac{\pi^2}{12} \sqrt{\zeta^2 - 1} \theta(\zeta - 1) + \int_0^\zeta dx \sqrt{\zeta^2 - x^2} f^{WK}(x) \right], \quad (16)$$

$$\begin{aligned} \Delta E^{WK}(2S) = & \frac{\mu\alpha(Z\alpha)^4}{2\pi} \int_0^\infty \frac{d\zeta}{\zeta^4} \left(\frac{1}{p_2^2(\zeta)} - \frac{2}{p_2^3(\zeta)} + \frac{3}{2p_2^4(\zeta)} \right) \times \\ & \times \left[-\frac{\pi^2}{12} \sqrt{\zeta^2 - 1} \theta(\zeta - 1) + \int_0^\zeta dx \sqrt{\zeta^2 - x^2} f^{WK}(x) \right]. \end{aligned} \quad (17)$$

Numerical values of Eqs. (16) and (17) are included in Table I.

Let us consider the modification of the Coulomb potential by virtue of two-loop vacuum polarization (see Fig.1(b,c,d)). The contribution of the diagram (b) in Fig.1 containing two sequential electronic loops can be obtained by applying the replacement (8) two times in the photon propagator. In the coordinate representation the required interaction operator of the particles has the form:

$$V_{1\gamma, VP-VP}^C(r) = \frac{\alpha^2}{9\pi^2} \int_1^\infty \rho(\xi) d\xi \int_1^\infty \rho(\eta) d\eta \left(-\frac{Z\alpha}{r} \right) \frac{1}{(\xi^2 - \eta^2)} \left(\xi^2 e^{-2m_e \xi r} - \eta^2 e^{-2m_e \eta r} \right). \quad (18)$$

It gives the following results for the energy spectrum:

$$\Delta E_{1\gamma, VP-VP}(1S) = -\frac{\mu\alpha^2(Z\alpha)^2}{9\pi^2} \int_1^\infty \rho(\xi) d\xi \int_1^\infty \rho(\eta) d\eta \frac{1}{(\xi^2 - \eta^2)} \left(\frac{\xi^2}{p_1^2(\xi)} - \frac{\eta^2}{p_1^2(\eta)} \right) = \quad (19)$$

$$= \begin{cases} \mu p : & -1.8816 \text{ meV} \\ \mu d : & -2.1871 \text{ meV}, \end{cases}$$

$$\begin{aligned} \Delta E_{1\gamma, VP-VP}(2S) = & -\frac{\mu\alpha^2(Z\alpha)^2}{18\pi^2} \int_1^\infty \rho(\xi) d\xi \int_1^\infty \rho(\eta) d\eta \frac{1}{(\xi^2 - \eta^2)} \times \\ & \times \left[\xi^2 \left(\frac{1}{p_2^2(\xi)} - \frac{2}{p_2^3(\xi)} + \frac{3}{2p_2^4(\xi)} \right) - \eta^2 \left(\frac{1}{p_2^2(\eta)} - \frac{2}{p_2^3(\eta)} + \frac{3}{2p_2^4(\eta)} \right) \right] = \begin{cases} \mu p : & -0.2426 \text{ meV} \\ \mu d : & -0.2811 \text{ meV}. \end{cases} \end{aligned} \quad (20)$$

The contributions of the diagrams (b,c) in Fig.1 which are determined by the second order polarization operator can be calculated after the following replacement in the photon propagator [33]:

$$\begin{aligned} \frac{1}{k^2} \rightarrow & \left(\frac{\alpha}{\pi} \right)^2 \int_0^1 \frac{f(v)}{4m_e^2 + k^2(1 - v^2)} dv = \left(\frac{\alpha}{\pi} \right)^2 \frac{2}{3} \int_0^1 dv \frac{v}{4m_e^2 + k^2(1 - v^2)} \times \\ & \times \left\{ (3 - v^2)(1 + v^2) \left[Li_2 \left(-\frac{1 - v}{1 + v} \right) + 2Li_2 \left(\frac{1 - v}{1 + v} \right) + \frac{3}{2} \ln \frac{1 + v}{1 - v} \ln \frac{1 + v}{2} - \ln \frac{1 + v}{1 - v} \ln v \right] + \right. \\ & \left. \left[\frac{11}{16}(3 - v^2)(1 + v^2) + \frac{v^4}{4} \right] \ln \frac{1 + v}{1 - v} + \left[\frac{3}{2}v(3 - v^2) \ln \frac{1 - v^2}{4} - 2v(3 - v^2) \ln v \right] + \frac{3}{8}v(5 - 3v^2) \right\}. \end{aligned} \quad (21)$$

The contribution value can then conveniently be calculated in the coordinate representation with the reduction of the interparticle interaction potential to the form:

$$\Delta V_{1\gamma, \text{2-loop VP}}^C = -\frac{2}{3} \frac{Z\alpha}{r} \left(\frac{\alpha}{\pi}\right)^2 \int_0^1 \frac{f(v)dv}{(1-v^2)} e^{-\frac{2m_e r}{\sqrt{1-v^2}}}. \quad (22)$$

The operator (22) gives the following corrections to the Lamb shift of S -levels in muonic hydrogen and deuterium:

$$\Delta E_{1\gamma, \text{2-loop VP}}(1S) = -\frac{2}{3\pi^2} \mu\alpha^2 (Z\alpha)^2 \frac{W^2}{m_e^2} \int_0^1 \frac{f(v)dv}{\left(1 + \frac{W\sqrt{1-v^2}}{m_e}\right)^2} = \begin{cases} \mu p : -12.6144 \text{ meV} \\ \mu d : -14.0141 \text{ meV} \end{cases} \quad (23)$$

$$\begin{aligned} \Delta E_{1\gamma, \text{2-loop VP}}(2S) &= -\frac{1}{12\pi^2} \mu\alpha^2 (Z\alpha)^2 \frac{W^2}{m_e^2} \int_0^1 \frac{f(v)dv}{\left(1 + \frac{W\sqrt{1-v^2}}{2m_e}\right)^2} \left[1 - \frac{2}{p_3(v)} + \frac{3}{2p_3^2(v)}\right] = \\ &= \begin{cases} \mu p : -1.4112 \text{ meV} \\ \mu d : -1.5606 \text{ meV} \end{cases}, \quad p_3(v) = \frac{1}{1 + \frac{2m_e}{W\sqrt{1-v^2}}}. \end{aligned} \quad (24)$$

Note that, as we determine contributions to the energy spectrum numerically, the corresponding results are given with an accuracy of 0.0001 meV.

III. THREE-LOOP VACUUM POLARIZATION IN ONE-PHOTON INTERACTION

In the fifth order over α there is the contribution of three-loop amplitudes of the vacuum polarization in the one-photon interaction (see diagrams (a)-(b) in Fig.3). The contribution of the diagram (a) in Fig.3 to the potential takes the form:

$$\begin{aligned} V_{VP-VP-VP}^C(r) &= -\frac{Z\alpha}{r} \frac{\alpha^3}{(3\pi)^3} \int_1^\infty \rho(\xi) d\xi \int_1^\infty \rho(\eta d\eta) \int_1^\infty \rho(\zeta) d\zeta \times \\ &\times \left[e^{-2m_e \zeta r} \frac{\zeta^4}{(\xi^2 - \zeta^2)(\eta^2 - \zeta^2)} + e^{-2m_e \xi r} \frac{\xi^4}{(\zeta^2 - \xi^2)(\eta^2 - \xi^2)} + e^{-2m_e \eta r} \frac{\eta^4}{(\xi^2 - \eta^2)(\zeta^2 - \eta^2)} \right]. \end{aligned} \quad (25)$$

The contribution of the diagram (b) in Fig.3 can be written also in the integral form using expressions (8) and (21):

$$V_{VP-2-loop VP}^C = -\frac{4\mu\alpha^3 (Z\alpha)}{9\pi^3} \int_1^\infty \rho(\xi) d\xi \int_1^\infty \frac{f(\eta)}{\eta} d\eta \frac{1}{r} \left[e^{-2m_e \eta r} \frac{\eta^2}{\eta^2 - \xi^2} - e^{-2m_e \xi r} \frac{\xi^2}{\eta^2 - \xi^2} \right]. \quad (26)$$

The corrections to the energy spectrum of the (μp) and (μd) atoms corresponding to the interaction operators (25) and (26) are the following:

$$\begin{aligned} \Delta E_{VP-VP-VP}(1S) &= -\frac{\mu\alpha^3 (Z\alpha)^2}{27\pi^3} \int_1^\infty \rho(\xi) d\xi \int_1^\infty \rho(\eta) d\eta \int_1^\infty \rho(\zeta) d\zeta \times \\ &\times \left[\frac{\xi^4}{(\xi^2 - \eta^2)(\xi^2 - \zeta^2)p_1^2(\xi)} + \frac{\eta^4}{(\eta^2 - \xi^2)(\eta^2 - \zeta^2)p_1^2(\eta)} + \frac{\zeta^4}{(\zeta^2 - \xi^2)(\zeta^2 - \eta^2)p_1^2(\zeta)} \right] = \end{aligned} \quad (27)$$

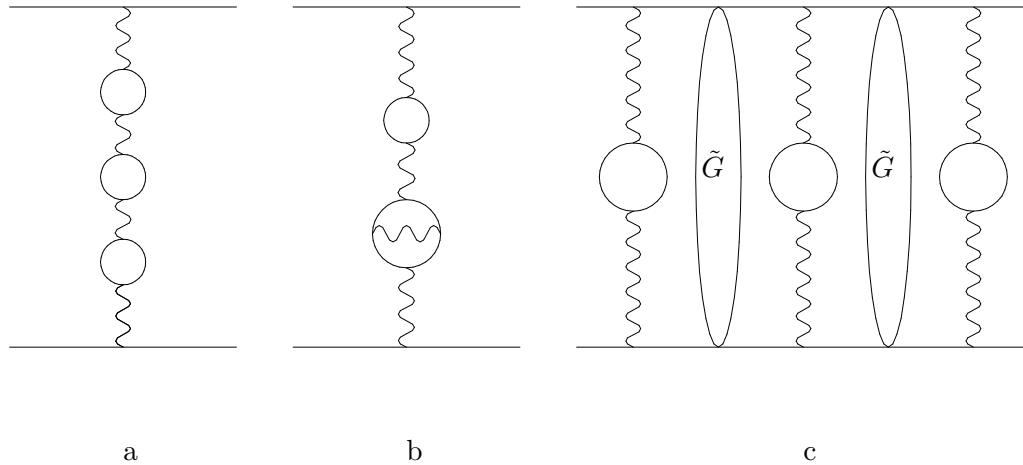


FIG. 3: Effects of three-loop vacuum polarization in one-photon interaction (a,b) and in the third order of the perturbation theory (c).

$$\begin{aligned}
 &= \begin{cases} \mu p : -0.0029 \text{ meV} \\ \mu d : -0.0034 \text{ meV} \end{cases} \\
 \Delta E_{VP-VP-VP}(2S) &= -\frac{\mu\alpha^3(Z\alpha)^2}{54\pi^3} \int_1^\infty \rho(\xi) d\xi \int_1^\infty \rho(\eta) d\eta \int_1^\infty \rho(\zeta) d\zeta \times \quad (28) \\
 &\times \left\{ \frac{\xi^4}{(\xi^2 - \eta^2)(\xi^2 - \zeta^2)} \left[\frac{1}{p_2^2(\xi)} - \frac{1}{p_2^3(\xi)} \right] + \frac{\eta^4}{(\eta^2 - \xi^2)(\eta^2 - \zeta^2)} \left[\frac{1}{p_2^2(\eta)} - \frac{1}{p_2^3(\eta)} \right] + \right. \\
 &\left. + \frac{\zeta^4}{(\zeta^2 - \xi^2)(\zeta^2 - \eta^2)} \left[\frac{1}{p_2^2(\zeta)} - \frac{1}{p_2^3(\zeta)} \right] \right\} = \begin{cases} \mu p : -0.0003 \text{ meV} \\ \mu d : -0.0004 \text{ meV} \end{cases} \\
 \Delta E_{VP-2-loop VP}(1S) &= -\frac{4\mu\alpha^3(Z\alpha)^2}{9\pi^3} \int_1^\infty \rho(\xi) d\xi \int_1^\infty \frac{f(\eta)}{\eta} \frac{d\eta}{\eta^2 - \xi^2} \left[\frac{\eta^2}{p_1^2(\eta)} - \frac{\xi^2}{p_1^2(\xi)} \right] \quad (29) \\
 &= \begin{cases} \mu p : -0.0223 \text{ meV} \\ \mu d : -0.0251 \text{ meV} \end{cases} \\
 \Delta E_{VP-2-loop VP}(2S) &= -\frac{2\mu\alpha^3(Z\alpha)^2}{9\pi^3} \int_1^\infty \rho(\xi) d\xi \int_1^\infty \frac{f(\eta)}{\eta} \frac{d\eta}{\eta^2 - \xi^2} \left\{ \eta^2 \left[\frac{1}{p_2^2(\eta)} - \frac{1}{p_2^3(\eta)} \right] - \right. \\
 &\left. - \xi^2 \left[\frac{1}{p_2^2(\xi)} - \frac{1}{p_2^3(\xi)} \right] \right\} = \begin{cases} \mu p : -0.0030 \text{ meV} \\ \mu d : -0.0035 \text{ meV} \end{cases} \quad (30)
 \end{aligned}$$

There is need to make the substitution $v = \sqrt{\eta^2 - 1}/\eta$ in the function $f(v)$ which is defined by Eq.(21). There exist also several diagrams which represent three-loop corrections to the polarization operator. They were calculated originally for the Lamb shift ($2P - 2S$) in the papers of the Kinoshita and Nio [34, 35]. The greatest contribution to the energy levels is determined by the diagrams of the sixth order vacuum polarization with the one electron loop (the contribution $\Pi^{(p6)}$ [34]). The estimation of their value to the $1S$ and $2S$ Lamb shift is presented in Table I.

IV. THE VACUUM POLARIZATION AND RELATIVISTIC EFFECTS

To calculate the energy spectra of the S -states of muonic hydrogen with a precision up to terms of order α^5 we have to construct the quasipotential on the basis of relations (5) and (6) which catch correctly the relativistic effects of necessary order (the Breit Hamiltonian ΔV_B). Accounting the electron vacuum polarization effects Pachucki has derived the Breit Hamiltonian ΔV_B^{VP} in Ref. [21]. Omitting the spin-dependent terms of the interaction operator let us write the two-body Breit Hamiltonians ΔV_B and ΔV_B^{VP} :

$$\Delta V_B = -\mathbf{p}^4 \left(\frac{1}{8m_1^3} + \frac{1}{8m_2^3} \right) + \frac{\pi Z\alpha}{2} \left(\frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \delta(\mathbf{r}) - \frac{Z\alpha}{2m_1 m_2 r} \left(\mathbf{p}^2 + \frac{r_i r_j p_i p_j}{r^2} \right), \quad (31)$$

$$\begin{aligned} \Delta V_B^{VP} = & \frac{\alpha}{3\pi} \int_1^\infty \rho(\xi) d\xi \left\{ \frac{Z\alpha}{2} \left(\frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \left[\pi \delta(\mathbf{r}) - \frac{m_e^2 \xi^2}{r} e^{-2m_e \xi r} \right] - \right. \\ & \left. - \frac{Z\alpha m_e^2 \xi^2}{m_1 m_2 r} e^{-2m_e \xi r} (1 - m_e \xi r) - \frac{Z\alpha}{2m_1 m_2} p_i \frac{e^{-2m_e \xi r}}{r} \left[\delta_{ij} + \frac{r_i r_j}{r^2} (1 + 2m_e \xi r) \right] p_j \right\}. \end{aligned} \quad (32)$$

In the first order of the perturbation theory (FOPT) the Hamiltonian (32) gives the following contribution after averaging over the Coulomb wave functions (11):

$$\Delta E_{1,B}^{rel,VP} = \langle \psi_{1S,2S} | \Delta V_B^{VP} | \psi_{1S,2S} \rangle = \begin{cases} \mu p(1S) : 0.1962 \text{ meV}; \mu p(2S) : 0.0249 \text{ meV} \\ \mu d(1S) : 0.2515 \text{ meV}; \mu d(2S) : 0.0322 \text{ meV} \end{cases} \quad (33)$$

These corrections are of order $\alpha(Z\alpha)^4$. Second-order perturbation theory (SOPT) corrections to the energy spectrum are determined by the reduced Coulomb Green function (RCGF) [36], whose partial expansion is written as:

$$\tilde{G}_n(\mathbf{r}, \mathbf{r}') = \sum_{l,m} \tilde{g}_{nl}(r, r') Y_{lm}(\mathbf{n}) Y_{lm}^*(\mathbf{n}'). \quad (34)$$

The radial function $\tilde{g}_{nl}(r, r')$ was obtained in Ref.[36] in the form of the Sturm expansion in the Laguerre polynomials. For the $1S$ and $2S$ -states these functions are the following:

$$\tilde{g}_{10}(r, r') = -4\mu^2 Z\alpha \left(\sum_{m=2}^{\infty} \frac{L_{m-1}^1(x) L_{m-1}^1(x')}{m(m-1)} + \frac{5}{2} - \frac{x}{2} - \frac{x'}{2} \right) e^{-\frac{x+x'}{2}}, \quad (35)$$

$$\tilde{g}_{20}(r, r') = -2\mu^2 Z\alpha \left[\sum_{m=1, m \neq 2}^{\infty} \frac{L_{m-1}^1(x) L_{m-1}^1(x')}{m(m-2)} + \left(\frac{5}{2} + x \frac{\partial}{\partial x} + x' \frac{\partial}{\partial x'} \right) L_1^1(x) L_1^1(x') \right] e^{-\frac{x+x'}{2}}, \quad (36)$$

where $x = \mu Z\alpha r$, L_n^m are the usual Laguerre polynomials defined as:

$$L_n^m(x) = \frac{e^x x^{-m}}{n!} \left(\frac{d}{dx} \right)^n (e^{-x} x^{n+m}). \quad (37)$$

As a several quasipotential terms contain $\delta(\mathbf{r})$, we must know $\tilde{G}_n(\mathbf{r}, 0)$. The corresponding expression for the reduced Coulomb Green function was obtained in Ref.[37] using the Hoestler representation for the Coulomb Green function and subtracting the pole term. This gave

$$\tilde{G}_{1S}(\mathbf{r}, 0) = \frac{Z\alpha \mu^2}{4\pi} \frac{2e^{-x/2}}{x} [2x(\ln x + C) + x^2 - 5x - 2], \quad (38)$$

$$\tilde{G}_{2S}(\mathbf{r}, 0) = -\frac{Z\alpha\mu^2}{4\pi} \frac{e^{-x/2}}{2x} [4x(x-2)(\ln x + C) + x^3 - 13x^2 + 6x + 4], \quad (39)$$

where $C = 0.5772\dots$ is the Euler constant. One-loop vacuum polarization gives the contribution in the second order of the perturbation theory which is specified by the relation:

$$\Delta E_n \text{ } SOPT = 2 \sum_{m=1, m \neq n}^{\infty} \frac{<\psi_n^C|\Delta V_B|\psi_m^C><\psi_m^C|V_{VP}^C|\psi_n^C>}{E_n^C - E_m^C}, \quad (40)$$

The matrix element of the operator \mathbf{p}^4 is expressed by means of the substitution $\mathbf{p}^4/4\mu^2 = (H_0 + Z\alpha/r)(H_0 + Z\alpha/r)$ ($H_0 = \mathbf{p}^2/2\mu - Z\alpha/r$). After that we employ the sequence of algebraic transformations:

$$\begin{aligned} & <\psi_n^C|\frac{Z\alpha}{r}H_0 \sum_{m, m \neq n} \frac{|\psi_m^C><\psi_m^C|}{E_n^C - E_m^C} V_{VP}^C|\psi_n^C> = \\ & = - <\psi_n^C|\frac{Z\alpha}{r} (I - |\psi_n^C><\psi_n^C|) V_{VP}^C|\psi_n^C> + E_n^C <\psi_n^C|\frac{Z\alpha}{r} \sum_{m, m \neq n} \frac{|\psi_m^C><\psi_m^C|}{E_n^C - E_m^C} V_{VP}^C|\psi_n^C>. \end{aligned} \quad (41)$$

Integrating the RCGF (34) over the coordinates r and r' we obtain the sums which are calculated accounting the range of a variable ξ ($1 \div \infty$) and $1/(1 + W/m_e\xi) < 1$. Typical matrix element for the $1S$ correction is the following:

$$\begin{aligned} I &= \int_0^\infty \left(1 - \frac{x}{2}\right) e^{-x} dx \int_0^\infty x' e^{-x' p_1(\xi)} dx' \left[\frac{5}{2} - \frac{x}{2} - \frac{x'}{2} + \sum \frac{L_{m-1}^1(x)L_{m-1}^1(x')}{m(m-1)} \right] = \\ &= \frac{5}{4p_1^2(\xi)} - \frac{1}{2p_1^3(\xi)} + \frac{\ln p_1(\xi)}{p_1^2(\xi)}, \\ & \sum_{m=2}^{\infty} \frac{[p_1(\xi) - 1]^{m-1}}{(m-1)p_1(\xi)^{m+1}} = \frac{1}{p_1^2(\xi)} \sum_{m=2}^{\infty} \frac{1}{(m-1)p_1^{m-1}(\xi)} = \frac{\ln p_1(\xi)}{p_1^2(\xi)}. \end{aligned} \quad (42)$$

Omitting other numerous intermediate analytical expressions we write the summary numerical value of the second order of the perturbation theory contribution (40) for the $1S$ and $2S$ -levels:

$$\Delta E_{2,}^{rel, VP} = \begin{cases} \mu p(1S) : -0.2644 \text{ meV}; \mu p(2S) : -0.0559 \text{ meV} \\ \mu d(1S) : -0.3194 \text{ meV}; \mu d(2S) : -0.0696 \text{ meV} \end{cases} \quad (43)$$

V. TWO-LOOP AND THREE-LOOP VACUUM POLARIZATION IN THE SECOND ORDER OF THE PERTURBATION THEORY

Two-loop vacuum polarization gives the correction in the second order of the perturbation theory (see the diagram (a) in Fig.4):

$$\Delta E_{SOPT}^{VP, VP} = <\psi_n^C|V_{VP}^C \cdot \tilde{G} \cdot V_{VP}^C|\psi_n^C>. \quad (44)$$

The calculation of this matrix element can be done with the relations (10), (35) and (36). As a result we obtain the corrections of order $\alpha^2(Z\alpha)^2$ for the $1S$ and $2S$ states:

$$\Delta E_{SOPT}^{VP, VP}(1S) = -\frac{\mu\alpha^2(Z\alpha)^2}{9\pi^2} \int_1^\infty \rho(\xi) d\xi \int_1^\infty \rho(\eta) d\eta \left[\frac{5}{2} \frac{1}{p_1^2(\xi)p_1^2(\eta)} - \right. \quad (45)$$

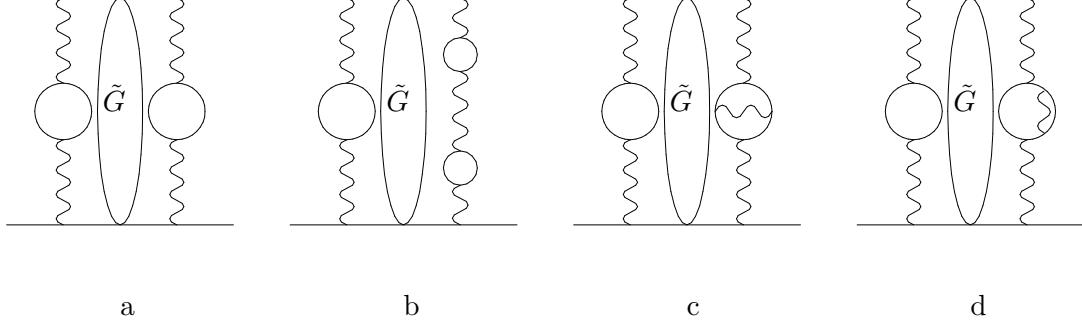


FIG. 4: Corrections of two-loop and three-loop vacuum polarization in the second order of the perturbation theory.

$$\begin{aligned}
& -\frac{1}{p_1^2(\xi)p_1^3(\eta)} - \frac{1}{p_1^3(\xi)p_1^2(\eta)} + \frac{[p_1(\xi) - 1][p_1(\eta) - 1]}{p_1^2(\xi)p_1^2(\eta)(p_1(\xi) + p_1(\eta) - 1)} - \frac{1}{p_1^2(\xi)p_1^2(\eta)} \ln \frac{[p_1(\xi) + p_1(\eta) - 1]}{(p_1(\xi) + p_1(\eta))} \Big], \\
& \Delta E_{SOPT}^{VP,VP}(2S) = -\frac{\mu\alpha^2(Z\alpha)^2}{9\pi^2} \int_1^\infty \rho(\xi)d\xi \int_1^\infty \rho(\eta)d\eta f(\xi, \eta), \quad (46) \\
& f(\xi, \eta) = -\frac{9}{2p_2^4(\xi)p_2^5(\eta)} + \frac{6}{p_2^3(\xi)p_2^5(\eta)} - \frac{3}{p_2^2(\xi)p_2^5(\eta)} - \frac{9}{2p_2^5(\xi)p_2^4(\eta)} + \frac{153}{8p_2^4(\xi)p_2^4(\eta)} - \frac{39}{2p_2^3(\xi)p_2^4(\eta)} + \\
& + \frac{33}{4p_2^2(\xi)p_2^4(\eta)} + \frac{6}{p_2^5(\xi)p_2^3(\eta)} - \frac{39}{2p_2^4(\xi)p_2^3(\eta)} + \frac{17}{p_2^3(\xi)p_2^3(\eta)} - \frac{6}{p_2^2(\xi)p_2^3(\eta)} - \frac{3}{p_2^5(\xi)p_2^2(\eta)} + \\
& + \frac{33}{4p_2^4(\xi)p_2^2(\eta)} - \frac{6}{p_2^3(\xi)p_2^2(\eta)} + \frac{3}{2p_2^2(\xi)p_2^2(\eta)} - \frac{z^4}{a_2^2(\xi)a_2^2(\eta)(z-1)} - \frac{z^4(6-5z)}{2a_2^2(\xi)a_2^2(\eta)(z-1)^2p_2(\xi)} - \\
& - \frac{z^4(6-5z)}{2a_2^2(\xi)a_2^2(\eta)(z-1)^2p_2(\eta)} + \frac{z^4(4-3z)}{2a_2^2(\xi)a_2^3(\eta)(z-1)^2} + \frac{z^4(4-3z)}{2a_2^3(\xi)a_2^2(\eta)(z-1)^2} + \frac{z^5(-30+49z-21z^2)}{4a_2^3(\xi)a_2^3(\eta)(z-1)^3} - \\
& - \frac{z^4(-18+27z-11z^2)}{4a_2^2(\xi)a_2^3(\eta)(z-1)^3p_2(\xi)} - \frac{z^4(-18+27z-11z^2)}{4a_2^3(\xi)a_2^2(\eta)(z-1)^3p_2(\eta)} + \frac{z^4(-10+13z-5z^2)}{4a_2^3(\xi)a_2^3(\eta)(z-1)^3} + \ln(1-z) \times \\
& \times \left[-\frac{2z^3}{a_2^2(\xi)a_2^2(\eta)} + \frac{3z^3}{a_2^2(\xi)a_2^2(\eta)p_2(\xi)} + \frac{3z^3}{a_2^2(\xi)a_2^2(\eta)p_2(\eta)} - \frac{z^3}{a_2^2(\xi)a_2^3(\eta)} - \frac{z^3}{a_2^3(\xi)a_2^2(\eta)} - \right. \\
& \left. \frac{9z^4}{2a_2^3(\xi)a_2^3(\eta)} + \frac{3z^3}{2a_2^2(\xi)a_2^3(\eta)p_2(\xi)} + \frac{3z^3}{2a_2^3(\xi)a_2^2(\eta)p_2(\eta)} - \frac{z^3}{2a_2^3(\xi)a_2^3(\eta)} \right],
\end{aligned}$$

where $a_2(\xi) = p_2(\xi) - 1$, $z = a_2(\xi)a_2(\eta)/p_2(\xi)p_2(\eta)$.

Three-loop vacuum polarization contributions in the second order of the perturbation theory shown in Fig.4(b,c,d) are of order $\alpha^3(Z\alpha)^2$. For their calculation we must use again the analytical expression for the one-loop and two-loop polarization operator. Fulfilling the integration over the wave function coordinates we represent the necessary corrections in the form:

$$\Delta E_{SOPT}^{VP-VP,VP}(1S) = -\frac{2\mu\alpha^3(Z\alpha)^2}{27\pi^3} \int_1^\infty \rho(\xi)d\xi \int_1^\infty \frac{\rho(\eta)d\eta}{(\xi^2 - \eta^2)} \int_1^\infty \frac{\rho(\zeta)d\zeta}{p_1^2(\zeta)} \left[\frac{\xi^2 g(\xi, \zeta)}{p_1^2(\xi)} - \frac{\eta^2 g(\eta, \zeta)}{p_1^2(\eta)} \right], \quad (47)$$

$$g(\xi, \zeta) = \frac{5}{2} - \frac{1}{p_1(\xi)} - \frac{1}{p_1(\zeta)} + \frac{[p_1(\xi) - 1][p_1(\zeta) - 1]}{p_1(\xi) + p_1(\zeta) - 1} - \ln \frac{p_1(\xi) + p_1(\zeta) - 1}{p_1(\xi) \cdot p_1(\zeta)}, \quad (48)$$

$$\Delta E_{SOPT}^{VP-VP,VP}(2S) = -\frac{2\mu\alpha^3(Z\alpha)^2}{27\pi^3} \int_1^\infty \rho(\xi)d\xi \int_1^\infty \rho(\eta)d\eta \int_1^\infty \rho(\zeta)d\zeta \frac{[\xi^2 f(\xi, \zeta) - \eta^2 f(\eta, \zeta)]}{\xi^2 - \eta^2}, \quad (49)$$

where the function $f(\xi, \zeta)$ is defined earlier by the Eq.(46),

$$\Delta E_{SOPT}^{2-loop VP,VP}(1S) = -\frac{\mu\alpha^3(Z\alpha)^2}{9\pi^2} \int_0^1 f(v)dv \left\{ \frac{5}{2} \frac{1}{p_1^2(\xi)q_1^2(v)} - \frac{1}{p_1^3(\xi)q_1^2(v)} + \right. \quad (50)$$

$$\left. + \frac{1}{p_1^2(\xi)q_1^2(v)} \left[\frac{p_1(\xi)q_1(v)}{p_1(\xi) + q_1(v) + 1} - \ln \frac{p_1(\xi) + q_1(v) + 1}{p_1(\xi) + q_1(v)} \right] \right\}, \quad q_1(v) = 1 + \frac{m_e}{W\sqrt{1-v^2}},$$

$$\Delta E_{SOPT}^{2-loop VP,VP}(2S) = -\frac{4\mu\alpha^3(Z\alpha)^2}{9\pi^3} \int_0^1 \frac{f(v)dv}{1-v^2} \int_1^\infty \rho(\xi)d\xi f(\xi, v), \quad (51)$$

where $p_2(v) = 1 + 2m_e/W\sqrt{1-v^2}$ in the function $f(\xi, v)$, $a_2(v) = 2m_e/W\sqrt{1-v^2}$. The energy terms (45)-(51) give the contribution to the (μp) - (μd) isotope shift:

$$\Delta E_{SOPT,IS}^{VP,VP} = 0.3125 \text{ meV}, \quad (52)$$

and the individual values for the S -states are depicted in Table I. The analysis of the three-loop vacuum polarization contribution in the third order of the perturbation theory (see the diagram (c) in Fig.3) shows that it is one order of the magnitude smaller than the contribution of the amplitude shown in Fig.4(c) and can be neglected.

VI. MUON SELF-ENERGY, MUON VACUUM POLARIZATION AND RECOIL CORRECTIONS

To estimate the corrections of the muon self-energy (MSE) and muon vacuum polarization (MVP) of order $\alpha(Z\alpha)^4$ we used well-known analytical results for the hydrogen atom. In the case of the $1S$ and $2S$ -states these contributions are the following [1]:

$$\Delta E_{MVP,MSE}(1S) = \frac{\alpha(Z\alpha)^4}{\pi} \frac{\mu^3}{m_1^2} \left[\frac{4}{3} \ln \frac{m_1}{\mu(Z\alpha)^2} - \frac{4}{3} \ln k_0(1, 0) + \frac{38}{45} \right] = \begin{cases} \mu p : 5.1180 \text{ meV} \\ \mu d : 5.9395 \text{ meV} \end{cases}, \quad (53)$$

$$\Delta E_{MVP,MSE}(2S) = \frac{\alpha(Z\alpha)^4}{8\pi} \frac{\mu^3}{m_1^2} \left[\frac{4}{3} \ln \frac{m_1}{\mu(Z\alpha)^2} - \frac{4}{3} \ln k_0(2, 0) + \frac{38}{45} \right] = \begin{cases} \mu p : 0.6543 \text{ meV} \\ \mu d : 0.7594 \text{ meV} \end{cases}, \quad (54)$$

where $\ln k_0(n, l)$ is the Bethe logarithm:

$$\ln k_0(1, 0) = 2.984128555765498, \quad \ln k_0(2, 0) = 2.811769893120563. \quad (55)$$

The correction determined as a sum of radiative insertions in the lepton line is known also in analytical form. We represent its value despite the fact that the order of this correction is $\alpha(Z\alpha)^5$ because the corresponding coefficient is sufficiently large (of order 10) [1]:

$$\Delta E_{rad}(1S) = \frac{\alpha(Z\alpha)^5 \mu^3}{m_1^2} \left(\frac{427}{96} - 2 \ln 2 \right) = \begin{cases} \mu p : 0.0355 \text{ meV} \\ \mu d : 0.0414 \text{ meV} \end{cases} \quad (56)$$

$$\Delta E_{rad}(2S) = \frac{\alpha(Z\alpha)^5 \mu^3}{8m_1^2} \left(\frac{427}{96} - 2 \ln 2 \right) = \begin{cases} \mu p : 0.0044 \text{ meV} \\ \mu d : 0.0052 \text{ meV} \end{cases}. \quad (57)$$

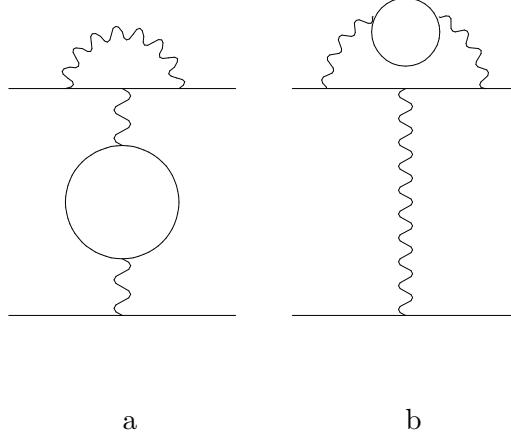


FIG. 5: Radiative corrections accounting electron vacuum polarization.

Following the Pachucki paper [21], let us estimate also the contributions of two diagrams of the sixth order over α enhanced by the $\ln \alpha$. The diagram (a) in Fig.5 of the muon radiative correction with electron vacuum polarization gives the following contributions:

$$\Delta E_{1,rad+VP}(1S) = \frac{2\mu^3\alpha^2(Z\alpha)^4}{9\pi^2m_1^2} \ln \frac{m_1}{\mu\alpha^2} \int_1^\infty \rho(\xi) d\xi \left[\frac{2}{p_1(\xi)} - \frac{2}{p_1^2(\xi)} + \frac{1}{p_1^3(\xi)} \right] = \quad (58)$$

$$= \begin{cases} \mu p : 0.0061 \text{ meV} \\ \mu d : 0.0073 \text{ meV} \end{cases}$$

$$\Delta E_{1,rad+VP}(2S) = \frac{\mu^3\alpha^2(Z\alpha)^4}{48\pi^2m_1^2} \ln \frac{m_1}{\mu\alpha^2} \int_1^\infty \rho(\xi) d\xi \times \quad (59)$$

$$\times \left[\frac{16}{3p_2(\xi)} - \frac{9}{2p_2^2(\xi)} + \frac{3}{p_2^3(\xi)} - \frac{9}{4p_2^4(\xi)} + \frac{1}{p_2^5(\xi)} \right] = \begin{cases} \mu p : 0.0010 \text{ meV} \\ \mu d : 0.0012 \text{ meV} \end{cases}$$

The diagram (b) in Fig.5 gives the contribution to the energy spectrum which can be expressed in terms of the slope of the Dirac form factor F'_1 and the Pauli form factor F_2 :

$$\Delta E_{2,rad+VP}(nS) = \frac{\mu^3}{m_1^2} \frac{(Z\alpha)^4}{n^3} \left[4m_1^2 F'_1(0) + F_2(0) \right], \quad (60)$$

where the form factors F'_1 and F_2 were calculated in Ref.[38]:

$$m_1^2 F'_1(0) = \left(\frac{\alpha}{\pi} \right)^2 \left[\frac{1}{9} \ln^2 \frac{m_1}{m_e} - \frac{29}{108} \ln \frac{m_1}{m_e} + \frac{1}{9} \zeta(2) + \frac{395}{1296} \right], \quad (61)$$

$$F_2(0) = \left(\frac{\alpha}{\pi} \right)^2 \left[\frac{1}{3} \ln \frac{m_1}{m_e} - \frac{25}{36} \right]. \quad (62)$$

Then the correction to the Lamb shift of the S-levels takes the form:

$$\Delta E_{2,rad+VP}(nS) = \frac{\mu^3\alpha^2(Z\alpha)^4}{m_1^2 n^3 \pi^2} \left(\frac{4}{9} \ln^2 \frac{m_1}{m_e} - \frac{20}{27} \ln \frac{m_1}{m_e} + \frac{4}{9} \zeta(2) + \frac{85}{162} \right). \quad (63)$$

In this section we included yet known analytical recoil correction of order $(Z\alpha)^5$ which is determined by two-photon exchange diagrams [39]:

$$\Delta E_{rec}(nS) = \frac{(Z\alpha)^5 \mu^3}{\pi n^3 m_1 m_2} \left[-\frac{2}{3} \ln(Z\alpha) - \frac{8}{3} \ln k_0(n, 0) - \frac{1}{9} - \frac{7}{3} a_n - \right. \\ \left. - \frac{2}{m_2^2 - m_1^2} \left(m_2^2 \ln \frac{m_1}{\mu} - m_1^2 \ln \frac{m_2}{\mu} \right) \right], \quad (64)$$

where

$$a_n = -2 \left[\ln \frac{2}{n} + \left(1 + \frac{1}{2} + \dots + \frac{1}{n} \right) + 1 - \frac{1}{2n} \right]. \quad (65)$$

Numerical values of the corrections (63) and (64) are presented in Table I.

VII. EFFECTS OF NUCLEAR STRUCTURE, POLARIZABILITY AND VACUUM POLARIZATION

In the energy spectrum of muonic hydrogen the important role belongs to strong interactions which are connected with the distributions of electric charge and magnetic moment of the nucleus. Namely these corrections together with the mass differences of the isotopes lead to the isotope shift in the system (μp) - (μd) . In the leading order $(Z\alpha)^4$ the nuclear structure effects are determined by the differential parameter of the electric charge distribution known as the nucleus charge radius r_N . To calculate one-loop corrections where the nucleus structure effects are essential we must know the form of the nucleus electromagnetic form factors. The contribution of the nuclear structure effects was studied both for the hyperfine structure and the Lamb shift in hydrogenic atoms in Refs.[1, 21, 22, 25, 40]. The leading order nuclear structure correction of order $(Z\alpha)^4$ to the Lamb shift of the S -levels in muonic hydrogen has the form (see the diagram (a) in Fig.6):

$$\Delta E_{str,(Z\alpha)^4}(nS) = \frac{2}{3n^3} \mu^3 (Z\alpha)^4 \langle r_p^2 \rangle, \quad (66)$$

where r_p^2 is the mean-square proton charge radius. Numerical values of the correction (66) for the levels with $n = 1$ and $n = 2$ (at $r_p = 0.891$ fm [1]) are presented in Table I. They give essential relative order contribution, so, the decrease of the error in the value of the proton charge radius is extremely important task if we desire to obtain more precise theoretical values of the Lamb shift of the S -states, different fine structure intervals and the isotope shifts. In the case of the muonic deuterium the deuteron charge radius $r_d = 2.094$ fm is used [25].

Two-photon exchange amplitudes shown in Fig.7 give the nuclear structure corrections of order $(Z\alpha)^5$. In this case the quasipotential of two-photon interaction can be found by means of the relations (5) and (6) [21, 22]. Corresponding correction to the energy spectrum takes the form of one-dimensional integral:

$$\Delta E_{str,(Z\alpha)^5}(nS) = -\frac{\mu^3 (Z\alpha)^5}{\pi n^3} \int_0^\infty \frac{dk}{k} V(k), \quad (67)$$

$$V(k) = \frac{2(F_1^2 - 1)}{m_1 m_2} + \frac{8m_1 [F_2(0) + 4m_2^2 F_1'(0)]}{m_2(m_1 + m_2)k} + \frac{k^2}{2m_1^3 m_2^3} \times \quad (68)$$

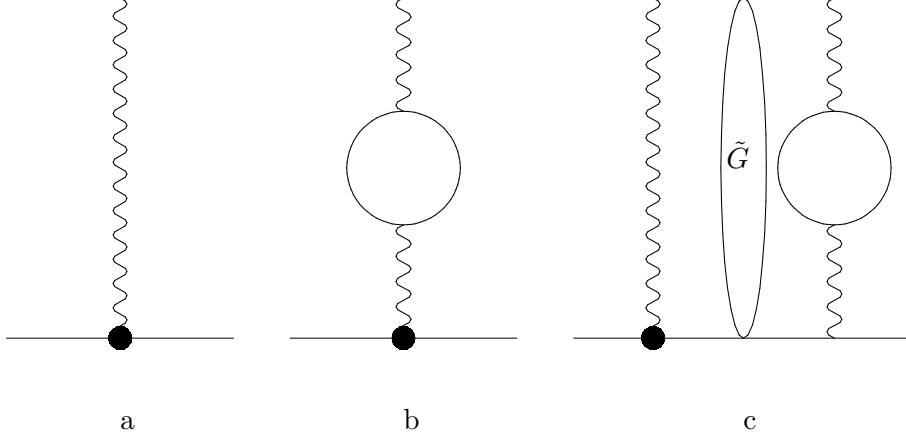


FIG. 6: The nuclear structure and vacuum polarization corrections.

$$\begin{aligned}
& \times \left[2(F_1^2 - 1)(m_1^2 + m_2^2) + 4F_1 F_2 m_1^2 + 3F_2^2 m_1^2 \right] + \frac{\sqrt{k^2 + 4m_1^2}}{2m_1^3 m_2 (m_1^2 - m_2^2) k} \times \\
& \times \left\{ k^2 \left[2(F_1^2 - 1)m_2^2 + 4F_1 F_2 m_1^2 + 3F_2^2 m_1^2 \right] - 8m_1^4 F_1 F_2 + \frac{16m_1^4 m_2^2 (F_1^2 - 1)}{k^2} \right\} - \\
& - \frac{\sqrt{k^2 + 4m_2^2} m_1}{2m_2^3 (m_1^2 - m_2^2) k} \left\{ k^2 \left[2(F_1^2 - 1) + 4F_1 F_2 + 3F_2^2 \right] - 8m_2^2 F_1 F_2 + \frac{16m_2^4 (F_1^2 - 1)}{k^2} \right\}.
\end{aligned}$$

To carry out numerical integration in Eq.(67) we used the parameterization of the Dirac F_1 and Pauli F_2 proton form factors obtained in Ref.[41]. For the muonic deuterium the similar contributions are obtained in Ref.[25]. The fifth order over α contribution connected with the nucleus structure is given also by the effects of the electron vacuum polarization which are shown in Fig.6(b,c). The particle interaction operator corresponding the amplitude in Fig.6(b) is determined by the following expression:

$$\Delta V_{str,VP}(r) = \frac{2\alpha(Z\alpha)r_N^2}{9} \int_1^\infty \rho(\xi) d\xi \left[\pi\delta(\mathbf{r}) - \frac{m_e^2 \xi^2}{r} e^{-2m_e \xi r} \right]. \quad (69)$$

The matrix elements of the operator (69) over the functions (11) lead to numerical data:

$$\Delta E_{str,VP}(1S) = \frac{2}{9} \alpha(Z\alpha)^4 \mu^3 r_N^2 \int_1^\infty \frac{\rho(\xi) d\xi}{p_1^2(\xi)} \left(1 + \frac{2m_e \xi}{W} \right) = \begin{cases} \mu p : 0.1991 \text{ meV} \\ \mu d : 1.4155 \text{ meV} \end{cases}, \quad (70)$$

$$\begin{aligned}
\Delta E_{str,VP}(2S) &= \frac{1}{36} \alpha(Z\alpha)^4 \mu^3 r_N^2 \int_1^\infty \frac{\rho(\xi) d\xi}{p_1^2(\xi)} \left[\left(1 + \frac{4m_e \xi}{W} \right) - \frac{4m_e^2 \xi^2}{W^2} \left(-\frac{2}{p_2(\xi)} + \frac{3}{2p_2^2(\xi)} \right) \right] = \\
&= \begin{cases} \mu p : 0.0257 \text{ meV} \\ \mu d : 0.1824 \text{ meV} \end{cases}.
\end{aligned} \quad (71)$$

There exist also the contribution of the electron vacuum polarization and nuclear structure in the second order of the perturbation theory (see the diagram (c) in Fig.6) which is

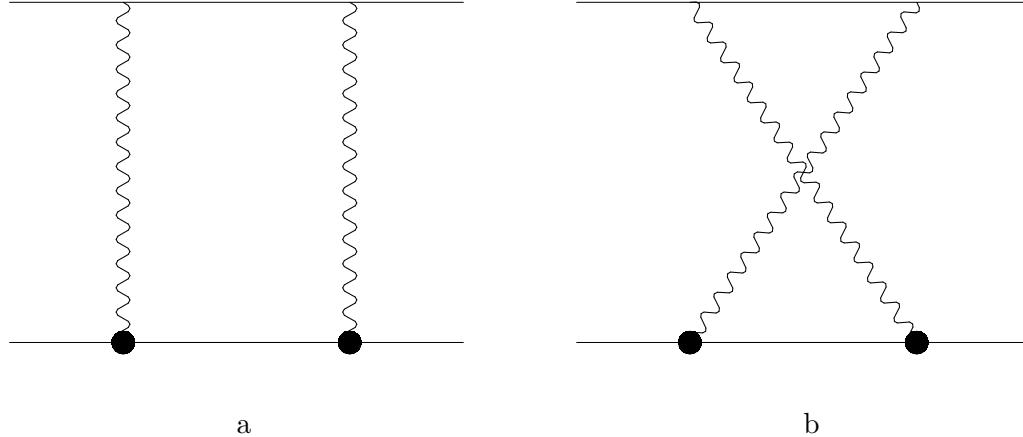


FIG. 7: The nuclear structure corrections of order $(Z\alpha)^5$. Bold circle on the diagram denotes the nucleus vertex operator.

determined by the reduced Coulomb Green function $\tilde{G}_n(r, 0)$ (38) and (39). In this case the contributions to the shifts of the S -levels are:

$$\begin{aligned} \Delta E_{str, VP; SOPT}(1S) &= -\frac{2}{9\pi}\alpha(Z\alpha)^4\mu^3r_N^2 \int_1^\infty \frac{\rho(\xi)d\xi}{p_1^3(\xi)} [2 - 3p_1(\xi) - 2p_1^2(\xi) - 2p_1(\xi)\ln p_1(\xi)] = \\ &= \begin{cases} \mu p : 0.1242 \text{ meV} \\ \mu d : 0.8913 \text{ meV} \end{cases}, \end{aligned} \quad (72)$$

$$\begin{aligned} \Delta E_{str, VP; SOPT}(2S) &= \frac{1}{36\pi}\alpha(Z\alpha)^4\mu^3r_N^2 \int_1^\infty \frac{\rho(\xi)d\xi}{p_1^5(\xi)} [-12 + 23p_2(\xi) - 8p_2^2(\xi) - 4p_2^3(\xi) + 4p_2^4(\xi) + \\ &\quad + 4p_2(\xi)(3 - 4p_2(\xi) + 2p_2^2(\xi))\ln p_2(\xi)] = \begin{cases} \mu p : 0.0126 \text{ meV} \\ \mu d : 0.0898 \text{ meV} \end{cases}. \end{aligned} \quad (73)$$

The strong interaction contribution to the energy spectrum of the (μp) and (μd) comes not only from the nucleus structure but also from the nucleus polarizability. For muonic deuterium atom we used analytical expression of the deuteron polarizability correction obtained in Ref.[42]:

$$\Delta E_{d,pol}(nS) = -\frac{\alpha(Z\alpha)^3}{\pi n^3}\mu^3m_1 \left[5\alpha_d \left(\ln \frac{8I}{m_1} + \frac{1}{20} \right) - \beta_d \left(\ln \frac{8I}{m_1} - 1.24 \right) \right], \quad (74)$$

where the deuteron bound energy $I = \kappa^2/m_2$, and $\kappa = 45.7 \text{ MeV}$, $\alpha_d = 0.635 \text{ fm}^3$ and $\beta_d = 0.073 \text{ fm}^3$ are the electric and magnetic deuteron polarizabilities. For muonic hydrogen atom the proton polarizability contribution to the Lamb shift of the S -states can be calculated employing the expression [22, 23]:

$$\begin{aligned} \Delta E_{p,pol}(nS) &= -\frac{16\mu^3(Z\alpha)^5m_1}{\pi^2n^3} \int_0^\infty \frac{dk}{k} \int_0^\pi d\phi \int_{\nu_0}^\infty dy \frac{\sin^2 \phi}{(k^2 + 4m_1^2 \cos^2 \phi)(y^2 + k^2 \cos^2 \phi)} \times \\ &\quad \times [(1 + 2\cos^2 \phi) \frac{(1 + k^2/y^2) \cos^2 \phi}{1 + R(y, k^2)} + \sin^2 \phi] F_2(y, k^2) + \frac{2\mu^3\alpha^5}{\pi n^3 m_1 m_2} \int_0^\infty h(k^2) \beta(k^2) k dk, \end{aligned} \quad (75)$$

where F_2 is the structure function of the lepton-nucleon scattering, $R = \sigma_L/\sigma_T$ is the ratio of the cross sections for the absorption of longitudinally and transversely polarized photons by hadrons, ν_0 is the threshold for the π -meson production. The subtraction term in Eq.(75) includes the function

$$h(k^2) = 1 + \left(1 - \frac{k^2}{2m_1^2}\right) \left(\sqrt{\frac{4m_1^2}{k^2} + 1} - 1\right), \quad \beta(k^2) = \beta \cdot G(k^2), \quad G(k^2) = \frac{1}{(1 + k^2/0.71)^2}, \quad (76)$$

where $\beta_p = 2.1(0.9) \times 10^{-4} \text{ fm}^3$ is the proton magnetic polarizability. The contribution of internal deuteron polarizability can be calculated also on the basis of Eq.(75) [43]. It is included in total value together with the correction (74) which is dominating.

Hadronic vacuum polarization contribution of order $\alpha(Z\alpha)^4$ to the shift of S -levels in muonic hydrogen which was studied in Ref.[44, 45] can be presented in the form:

$$\Delta E_{HVP}(nS) = -\frac{4\alpha(Z\alpha)^4\mu^3}{\pi n^3} \int_{4m_\pi^2}^{\infty} \frac{\rho^h(s)ds}{s}, \quad (77)$$

where the spectral function $\rho^h(s)$ is expressed through the cross section of e^+e^- annihilation into hadrons:

$$\rho^h(s) = \frac{\sigma^h(e^+e^- \rightarrow \text{hadrons})}{3s\sigma_{\mu\mu}(e^+e^- \rightarrow \mu^+\mu^-)}. \quad (78)$$

Dividing the whole integration region over s into the intervals where the cross section of e^+e^- -annihilation into hadrons is known from the experiment [46] we can carry out numerical integration in Eq.(77). Total numerical results are included in Table I.

VIII. CONCLUSION

In this paper we calculate different quantum electrodynamic corrections, effects of the nucleus structure and polarizability, hadronic vacuum polarization to the Lamb shift of the $1S$ and $2S$ energy levels in muonic hydrogen, muonic deuterium and isotope shift (μp) - (μd) for the splitting ($1S - 2S$). The main goal of the work is to summarize various corrections to the fine structure interval ($1S - 2S$) in the (μp), (μd) and the muonic hydrogen - muonic deuterium isotope shift for this splitting which would allow to obtain reliable theoretical values for the indicated quantities. We consider the corrections of orders α^3 , α^4 , α^5 and also several important contributions of order α^6 enhanced by the $\ln \alpha$. In our calculation we took into account that the ratio $\mu\alpha/m_e$ is close to one and centred special attention on the effects of the electron vacuum polarization. Numerical values of obtained contributions are presented in Table I. We included also in Table I the references on many papers where analytical or numerical calculations of some corrections are performed despite the fact that particular numerical results for the $1S$ and $2S$ -levels are absent in these papers. For the comparison of obtained results with the calculations carried out by other authors we used commonly the review article [1] which accumulate recent advances in the physics of the energy spectra of simple atomic systems and contain detailed references on the earlier performed investigations.

Let us point out some peculiarities of performed calculations:

1. The effects of the vacuum polarization for the considered muonic atoms have played significant role. They lead to the modification of the Breit two-particle interaction operator and give the corrections in the energy spectra up to fifth order in α .

2. The nuclear structure effects in the energy spectrum of the S -states are expressed in terms of the nucleus (the proton and deuteron) charge radius in the leading order $(Z\alpha)^4$ and in the next to leading order $(Z\alpha)^5$ for the one-loop amplitudes by means of the nucleus electromagnetic form factors.

3. The estimation of the nuclear polarizability contributions is performed on the basis of Eq.(74) obtained by the Mil'shtein, Petrosyan and Khriplovich and Eq.(75) in Refs.[22, 23]. The nuclear structure and polarizability contributions lead to largest theoretical uncertainty for the transition $(1S - 2S)$ and the isotope shift $(\mu p) - (\mu d)$.

Total numerical values for the energy levels $1S$ and $2S$ in muonic hydrogen and muonic deuterium presented in Table I and also the values of the fine structure interval $(1S - 2S)$, isotope shift $(\mu p) - (\mu d)$ can be considered as a proper estimation for the future experiments with these muonic atoms. The values of the corrections are obtained with the accuracy 0.0001 meV. For the splitting $(1S - 2S)$ the error of theoretical result is determined by a number of the factors. The uncertainties of fundamental parameters (fine structure constant, the proton, deuteron and muon masses) amount the value of order 10^{-7} . The quantum electrodynamic corrections of higher order α^6 give theoretical error of order 10^{-8} . The largest contribution to the theoretical error is connected with the uncertainties of the proton and deuteron charge radii. Their relative contributions reaches the value of order 10^{-6} (we used the values of the proton and deuteron charge radii: $r_p = 0.891$ fm, $r_d = 2.094$ fm). Further improvements of theoretical results presented in Table I are related primarily with the nuclear structure and polarizability corrections. So, for the comparison of performed calculations with future experimental data a major interest is connected with the quantity of fine structure interval $[E(1S) - 8E(2S)]$ which has not contain the nuclear structure and polarizability corrections of the leading order $(Z\alpha)^4$. Numerical values for this fine structure interval in muonic hydrogen and muonic deuterium are the following:

$$\mu p : E(1S) - 8E(2S) = -7\ 307\ 509\ 956\ 602.3099 \text{ meV}, \quad (79)$$

$$\mu d : E(1S) - 8E(2S) = -13\ 868\ 895\ 169\ 916.6917 \text{ meV}.$$

The relative value of that part of theoretical error in Eq.(79) which is determined by the QED corrections of higher order is extremely small (of order 10^{-15}).

The muonic hydrogen - muonic deuterium isotope shift for the splitting $(1S - 2S)$ is considered to be among the most important characteristics of the energy spectra of muonic atoms. The differences in wavelengths of lines emitted by isotopes of the same element can arise either as a result of the differences in the masses of the isotopes or on account of differences in the nuclear charge distributions. Whereas the masses of the proton and deuteron are determined at present with sufficiently high accuracy, the nuclear structure parameters are known less precisely. Using the results of fulfilled calculations we can express the difference of the deuteron and proton charge radii in terms of isotope shift:

$$\frac{r_d^2}{\left(1 + \frac{m_\mu}{m_d}\right)^3} - \frac{r_p^2}{\left(1 + \frac{m_\mu}{m_p}\right)^3} = \frac{12}{7m_\mu^3(Z\alpha)^4} \left(\Delta\tilde{E}_{IS}^{th} - \Delta E_{IS}^{exp} \right), \quad (80)$$

where theoretical value $\Delta\tilde{E}_{IS}^{th}$ doesn't contain the nuclear structure correction of order $(Z\alpha)^4$. So, the measurement of the isotope shift $(\mu p) - (\mu d)$ will allow on the one hand to perform additional test of the QED and on the other hand to obtain more accurate value of the deuteron charge radius from the relation (80) after the extraction the proton charge radius in the experiment at Paul Scherrer Institute (PSI) [4].

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TABLE I: Corrections of order α^3 , α^4 and α^5 to the $1S$ and $2S$ Lamb shifts in muonic hydrogen and muonic deuterium and isotope shift ΔE_{IS} .

The contribution to the energy of the atom	μp , meV		μd , meV		ΔE_{IS}	Ref.
	1S	2S	1S	2S		
	2	3	4	5	6	7
Fine structure formula $E_n = m_1 + m_2 - \frac{\mu^2(Z\alpha)^2}{2n^2} - \frac{\mu(Z\alpha)^4}{2n^3} \left[1 - \frac{3}{4n} + \frac{\mu^2}{4nm_1m_2} \right]$	1043927826 ~ 470.3586	1043929722 ~ 866.0601	1981268455 ~ 762.7537	1981270453 ~ 188.8081	101030.3530	(3) [1]
One-loop VP in 1γ interaction of order $\alpha(Z\alpha)^2$	-1898.8379	-219.5849	-2129.2820	-245.3205	204.7085	(12) (13) (14)
Wichmann-Kroll correction of order $\alpha(Z\alpha)^4$	0.0114	0.0012	0.0126	0.0014	-0.0010	(16) (17) [31]
Two-loop VP (VP-VP) in 1γ interaction of order $\alpha^2(Z\alpha)^2$	-1.8816	-0.2426	-2.1871	-0.2811	0.2616	(19) (20)
Two-loop VP (2-loop) in 1γ interaction of order $\alpha^2(Z\alpha)^2$	-12.6144	-1.4112	-14.0141	-1.5606	1.2476	(23) (24)
Three-loop VP (VP- VP-VP) in 1γ interaction of order $\alpha^3(Z\alpha)^2$	-0.0029	-0.0003	-0.0034	-0.0004	0.0004	[34] (27) (28)
Three-loop VP (VP- 2-loop) in 1γ interaction of order $\alpha^3(Z\alpha)^2$	-0.0223	-0.0030	-0.0251	-0.0035	0.0023	[34] (29) (30)
Three-loop VP ($\Pi^{(p6)}$) in 1γ interaction of order $\alpha^3(Z\alpha)^2$	-0.0340	-0.0045	-0.0380	-0.0050	0.0035	[34]
Relativistic and VP correction in FOPT of order $\alpha^3(Z\alpha)^2$	0.1962	0.0249	0.2515	0.0322	-0.0480	[21] (33)
Relativistic and VP effects in SOPT of order $\alpha^3(Z\alpha)^2$	-0.2644	-0.0559	-0.3194	-0.0696	0.0413	[21] (43)
Two-loop VP in SOPT of order $\alpha^2(Z\alpha)^2$	-2.0343	-0.1532	-2.3675	-0.1750	0.3114	(45) (46)
Three-loop VP (VP-VP,VP) in SOPT of order $\alpha^3(Z\alpha)^2$	-0.0061	-0.0002	-0.0073	-0.0005	0.0009	(47) (49)

Table I (continued).

1	2	3	4	5	6	7
Three-loop VP (2-loop VP,VP) in SOPT of order $\alpha^3(Z\alpha)^2$	-0.0059	-0.0016	-0.0069	-0.0021	0.0005	(50) (51)
Muon SE and VP of order $\alpha(Z\alpha)^4$	5.1180	0.6543	5.9395	0.7594	-0.7164	(53) (54)
Radiative corrections of order $\alpha(Z\alpha)^5$	0.0355	0.0044	0.0414	0.0052	-0.0051	(56) (57) [1]
Radiative and VP corrections of order $\alpha^2(Z\alpha)^4$	0.0178	0.0025	0.0209	0.0029	-0.0027	(58) (59) (63),[21]
Recoil correction of order $(Z\alpha)^5$	0.3009	0.0428	0.1781	0.0253	0.1053	(64) [1]
Nuclear structure correction of order $(Z\alpha)^4$	38.5711	4.8214	213.4218	26.6825	-152.6597	(66) [1, 25]
Nuclear structure correction of order $(Z\alpha)^5$	-0.1464	-0.0183	-2.9384	-0.3674	2.4429	(69) [21, 25]
Nuclear structure and VP correction of order $\alpha(Z\alpha)^4$	0.2127	0.0274	1.4155	0.1824	-1.0478	(70) (71)
Nuclear structure and VP correction in SOPT of order $\alpha(Z\alpha)^4$	0.1327	0.0135	0.8913	0.0898	-0.6823	(72) (73)
Nuclear polarizability correction of order $(Z\alpha)^5$	-0.1291	-0.0161	92.0511	11.5064	-80.6577	(74) (75) [42] [40, 43]
HVP correction of order $\alpha(Z\alpha)^4$	-0.0864	-0.0108	-0.1010	-0.0126	0.0128	[44] [45] (77)
Summary contribution	1043927824 ~598.8893	1043929722 ~650.1499	1981268453 ~925.6873	1981270452 ~980.2974	101 003.3495	